Some Analytic or Asymptotic Confluent Expansions for Functions of Several Variables

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Abstract. This paper aims at presenting multivariable extensions of the recent results of J. L. Fields [5] and others (cf. [1]–[4]) on certain analytic or asymptotic confluent expansions for functions of one and two variables. It is also demonstrated how these extensions would apply, for instance, to derive an asymptotic confluent expansion for a certain class of the generalized Lauricella function of several variables.

1. **Introduction.** In recent years several writers have contributed to the theory of analytic or asymptotic confluent expansions for functions of one and two variables (see, for instance, [1]–[5]; see also [6, pp. 50–57]). The object of the present paper is to discuss extensions of these results to certain functions of several variables. In order to illustrate the usefulness of these extensions, an asymptotic confluent expansion for a certain class of the generalized Lauricella function [7, p. 454 *et seq.*] is obtained. It is also shown how this last expansion (4.7) below can be further specialized to yield the corresponding asymptotic confluent expansion for the ordinary Lauricella functions $F_A^{(r)}$, $F_B^{(r)}$ and $F_D^{(r)}$ of r variables.

The following Tricomi-Erdélyi result (cf., e.g., [6, p. 33]) will be required in our investigation:

$$(1.1) \qquad \frac{\Gamma(z + \alpha)}{\Gamma(z + \beta)} = z^{\alpha-\beta} \sum_{j=0}^{N-1} \frac{(-1)^j (\beta - \alpha)_j}{j!} B_j^{(\alpha-\beta+1)}(\alpha) z^{-j} + z^{\alpha-\beta} O(z^{-N}),$$

$$|\arg(z + \alpha)| \leqslant \pi - \epsilon, \epsilon > 0,$$

where α and β are bounded complex numbers, and $B_j^{(\alpha-\beta+1)}(\alpha)$ is the generalized Bernoulli polynomial.

2. **Analytic confluent expansions.** Our first result is the analytic confluent expansion (2.3) contained in

THEOREM 1. *Let*

$$(2.1) \qquad \sum_{k_1,\ldots,k_r=0}^{\infty} a(k_1,\ldots,k_r) z_1^{k_1} \cdots z_r^{k_r} < \infty, \qquad |z_i| < R_i, i = 1,\ldots,r.$$

Then[**]

$$(2.2) \quad F[z_i; \mu_i; \sigma_i] = \sum_{k_1,\ldots,k_r=0}^{\infty} a(k_1,\ldots,k_r) \prod_{i=1}^{r} \frac{(\sigma_i)_{\mu_i k_i}}{(k_i!)^{\mu_i}} \left(\frac{z_i}{\sigma_i}\right)^{k_i} < \infty,$$

provided $|z_i/\sigma_i| < \mu_i^{-\mu_i} R_i$, $\mu_i > 0$, $i = 1,\ldots,r$; *and* $F(z_1,\ldots,z_r: \sigma)$ *can be re-arranged in the form,*

$$(2.3) \quad F(z_1,\ldots,z_r: \sigma) = \sum_{j_1,\ldots,j_r=0}^{\infty} g(j_1,\ldots,j_r; z_1,\ldots,z_r)\sigma^{-j_1-\cdots-j_r},$$

where

$$|z_i/\sigma| < R_i, \quad i = 1,\ldots,r, \quad F(z_1,\ldots,z_r: \sigma) = F[z_i; 1; \sigma],$$

and $g(j_1,\ldots,j_r; z_1,\ldots,z_r)$ *are given explicitly by* (2.9) *below.*

Further, for $j_1,\ldots,j_r \geqslant 1$, *the coefficients,* $g(j_1,\ldots,j_r; z_1,\ldots,z_r)$, *can be expressed in terms of the derivatives of*

$$(2.4) \qquad g(z_1,\ldots,z_r) = g(0,\ldots,0; z_1,\ldots,z_r).$$

Proof. From (2.1) and the test for absolute convergence of multiple series given, for instance, by Srivastava and Daoust [8, p. 157 *et seq.*], it is easily verified that the series for $F[z_i; \mu_i; \sigma_i]$ in (2.2) converges absolutely when $|z_i/\sigma_i| < \mu_i^{-\mu_i} R_i$, $i = 1,\ldots,r$.

For $\sigma \neq 0$, and integers $k_i \geqslant 0$, $i = 1,\ldots,r$, we have

$$(2.5) \quad \frac{(\sigma)_{k_i}}{\sigma^{k_i}} = \left(1 + \frac{k_i}{\sigma}\right)^{-1} \prod_{j_i=0}^{k_i} \left(1 + \frac{j_i}{\sigma}\right) = \sum_{j_i=0}^{k_i} \alpha_i(j_i, k_i)\sigma^{-j_i}, \quad \text{say,}$$

where $\alpha_i(j_i, k_i) > 0$, $i = 1,\ldots,r$.

Substitution in (2.2) with $\mu_i = 1$ and $\sigma_i = 1$, $i = 1,\ldots,r$, will show that $F(z_1,\ldots,z_r: \sigma)$ is majorized by the convergent series,

$$(2.6) \quad \begin{aligned} &\sum_{k_1,\ldots,k_r=0}^{\infty} \frac{|a(k_1,\ldots,k_r)||z_1|^{k_1} \cdots |z_r|^{k_r}}{k_1! \cdots k_r!} \\ &\quad \cdot \sum_{j_1=0}^{k_1} \cdots \sum_{j_r=0}^{k_r} \prod_{i=1}^{r} \{\alpha_i(j_i, k_i)\} |\sigma|^{-j_1-\cdots-j_r} \\ &= \sum_{k_1,\ldots,k_r=0}^{\infty} \frac{|a(k_1,\ldots,k_r)|(|\sigma|)_{k_1} \cdots (|\sigma|)_{k_r}}{k_1! \cdots k_r!} \left|\frac{z_1}{\sigma}\right|^{k_1} \cdots \left|\frac{z_r}{\sigma}\right|^{k_r}, \end{aligned}$$

where $|z_i/\sigma| < R_i$, $i = 1,\ldots,r$.

Thus $F(z_1,\ldots,z_r: \sigma)$ can be rearranged in descending powers of σ. Moreover,

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[**] Throughout this paper we abbreviate, for convenience, functions like $F[z_1,\ldots,z_r; \mu_1,\ldots,\mu_r; \sigma_1,\ldots,\sigma_r]$ by $F[z_i; \mu_i; \sigma_i]$.

since σ is arbitrary, the coefficients $g(j_1, \ldots, j_r; z_1, \ldots, z_r)$ will converge for arbitrary $z_1, \ldots, z_r$.

In order to identify the $\alpha_i(j_i, k_i)$ in (2.5), we compare (2.5) and the Tricomi-Erdélyi formula (1.1) with $z = \sigma$, $\alpha = k_i$ and $\beta = 0$; we thus obtain

$$(2.7) \qquad \alpha_i(j_i, k_i) = (1 - k_i)_{j_i} B_{j_i}^{(k_i)}/j_i!, \qquad i = 1, \ldots, r,$$

where $B_n^{(a)} = B_n^{(a)}(0)$ are the generalized Bernoulli numbers. Notice that

$$(2.8) \qquad \alpha_i(j_i, k_i) = \begin{cases} 1, & \text{if } j_i = 0, \, k_i \geqslant 0, \\ 0, & \text{if } j_i \geqslant 1, \, k_i = 0 \text{ or } k_i = 1, \end{cases}$$

for $i = 1, \ldots, r$.

From (2.3), (2.5) and (2.7) it follows that

$$
\begin{aligned}
g(j_1, \ldots, & j_r; z_1, \ldots, z_r) \\
(2.9) \qquad &= \sum_{k_1, \ldots, k_r = 0}^{\infty} a(k_1, \ldots, k_r) \prod_{i=1}^{r} \frac{(1 - k_i)_{j_i} z_i^{k_i}}{j_i! k_i!} B_{j_i}^{(k_i)}.
\end{aligned}
$$

For $k_i, j_i \geqslant 1$, $k_i^{-1} B_{j_i}^{(k_i)}$ is a polynomial of degree $(j_i - 1)$ in k_i. Thus, if Δ_x denotes the forward difference operator with respect to x, i.e., $\Delta_x \{f(x)\} = f(x + 1) - f(x)$, we have for $j_i \geqslant 1$, $i = 1, \ldots, r$,

$$(2.10) \qquad B_{j_i}^{(k_i)} = k_i \sum_{m_i = 0}^{j_i - 1} \frac{(-1)^{m_i} (1 + j_i - k_i)_{m_i}}{m_i!} \phi(m_i, j_i),$$

where, for convenience,

$$(2.11) \qquad \phi(m_i, j_i) = [\Delta_{k_i}^{m_i} \{B_{j_i}^{(k_i)} / k_i\}]_{k_i = j_i + 1} \qquad (i = 1, \ldots, r).$$

In view of (2.4), (2.9) and (2.10), it follows fairly easily that for $j_i \geqslant 1$, $i = 1, \ldots, r$,

$$
\begin{aligned}
g(j_1, \ldots, & j_r; z_1, \ldots, z_r) \\
(2.12) \qquad &= \sum_{m_1 = 0}^{j_1 - 1} \cdots \sum_{m_r = 0}^{j_r - 1} (-1)^J \frac{\partial^{r + J + m}}{\partial z_1^{1 + j_1 + m_1} \cdots \partial z_r^{1 + j_r + m_r}} \\
& \{g(z_1, \ldots, z_r)\} \prod_{i=1}^{r} \frac{z_i^{1 + j_i + m_i} \phi(m_i, j_i)}{j_i! m_i!},
\end{aligned}
$$

where the $\phi(m_i, j_i)$ are given by (2.11), and

$$(2.13) \qquad J = j_1 + \cdots + j_r, \qquad M = m_1 + \cdots + m_r.$$

This evidently completes the proof of Theorem 1.

Remark 1. A markedly different form of Theorem 1 may be given, for instance, by replacing the product,

$$\prod_{i=1}^{r} \frac{(\sigma_i)_{\mu_i k_i}}{(k_i!)^{\mu_i}} \left(\frac{z_i}{\sigma_i}\right)^{k_i},$$

in (2.2) by

$$(\sigma)_{\mu_1 k_1 + \cdots + \mu_r k_r} \prod_{i=1}^{r} \frac{(z_i/\sigma)^{k_i}}{(k_i!)^{\mu_i}}, \quad \mu_i > 0.$$

Remark 2. If we put

$$(2.14) \quad A_{m_1,\ldots,m_r}(z_1,\ldots,z_r) = z_1^{m_1} \cdots z_r^{m_r} \frac{\partial^{m_1+\cdots+m_r}}{\partial z_1^{m_1} \cdots \partial z_r^{m_r}} \{g(z_1,\ldots,z_r)\},$$

where

$$g(z_1,\ldots,z_r) = g(0,\ldots,0; z_1,\ldots,z_r)$$

$$(2.15)$$

$$= \sum_{k_1,\ldots,k_r=0}^{\infty} a(k_1,\ldots,k_r) \frac{z_1^{k_1}}{k_1!} \cdots \frac{z_r^{k_r}}{k_r!},$$

then the first few coefficients $g(j_1,\ldots,j_r; z_1,\ldots,z_r)$, given by (2.9) and (2.12), are as follows:

$$g(1,0,\ldots,0; z_1,\ldots,z_r) = \frac{1}{2} A_{2,0,\ldots,0}(z_1,\ldots,z_r),$$

$$\vdots$$

$$g(0,\ldots,0,1; z_1,\ldots,z_r) = \frac{1}{2} A_{0,\ldots,0,2}(z_1,\ldots,z_r);$$

$$g(1,1,0,\ldots,0; z_1,\ldots,z_r) = \frac{1}{4} A_{2,2,0,\ldots,0}(z_1,\ldots,z_r),$$

$$\vdots$$

$$g(0,\ldots,0,1,1; z_1,\ldots,z_r) = \frac{1}{4} A_{0,\ldots,0,2,2}(z_1,\ldots,z_r),$$

$$\vdots$$

$$g(1,\ldots,1; z_1,\ldots,z_r) = \frac{1}{2^r} A_{2,\ldots,2}(z_1,\ldots,z_r);$$

$$g(2,0,\ldots,0; z_1,\ldots,z_r)$$
$$= \frac{1}{3} A_{3,0,\ldots,0}(z_1,\ldots,z_r) + \frac{1}{8} A_{4,0,\ldots,0}(z_1,\ldots,z_r),$$

$$\vdots$$

$$g(0,\ldots,0,2; z_1,\ldots,z_r)$$
$$= \frac{1}{3} A_{0,\ldots,0,3}(z_1,\ldots,z_r) + \frac{1}{8} A_{0,\ldots,0,4}(z_1,\ldots,z_r);$$

$$g(1,2,0,\ldots,0; z_1,\ldots,z_r)$$
$$= \frac{1}{6} A_{2,3,0,\ldots,0}(z_1,\ldots,z_r) + \frac{1}{16} A_{2,4,0,\ldots,0}(z_1,\ldots,z_r);$$

$$g(2,1,0,\ldots,0; z_1,\ldots,z_r)$$
$$= \frac{1}{6} A_{3,2,0,\ldots,0}(z_1,\ldots,z_r) + \frac{1}{16} A_{4,2,0,\ldots,0}(z_1,\ldots,z_r);$$

$$g(3, 0, \ldots, 0; z_1, \ldots, z_r) = \frac{1}{4} A_{4,0,\ldots,0}(z_1, \ldots, z_r)$$

$$+ \frac{1}{6} A_{5,0,\ldots,0}(z_1, \ldots, z_r) + \frac{1}{48} A_{6,0,\ldots,0}(z_1, \ldots, z_r),$$

et cetera.

Another analytic confluent expansion for functions of several variables is given by

THEOREM 2. *Let*

$$(2.16) \qquad \sum_{k_1,\ldots,k_r=0}^{\infty} b(k_1, \ldots, k_r) z_1^{k_1} \cdots z_r^{k_r} < \infty, \qquad |z_i| < R_i, i = 1, \ldots, r.$$

Then

$$G[z_i; \lambda_i; \mu_i; \nu_i; \rho_i] = \sum_{k_1,\ldots,k_r=0}^{\infty} b(k_1, \ldots, k_r)$$

$$(2.17)$$

$$\cdot \prod_{i=1}^{r} \frac{(-\nu_i)_{\mu_i k_i} (\nu_i + \lambda_i)_{\rho_i k_i}}{(k_i!)^{\mu_i + \rho_i}} \left[- \frac{z_i}{\nu_i(\nu_i + \lambda_i)} \right]^{k_i} < \infty,$$

provided $|z_i| < |\nu_i(\nu_i + \lambda_i)| \mu_i^{-\mu_i} \rho_i^{-\rho_i} R_i$, $\mu_i > 0$, $\rho_i > 0$, $i = 1, \ldots, r$; *and*
$G(z_1, \ldots, z_r : \nu, \lambda)$ *can be rearranged in descending powers of* $\nu(\nu + \lambda)$ *to yield*

$$G(z_1, \ldots, z_r : \nu, \lambda)$$

$$(2.18)$$

$$= \sum_{j_1,\ldots,j_r=0}^{\infty} h(j_1, \ldots, j_r; z_1, \ldots, z_r; \lambda) \left[- \nu(\nu + \lambda) \right]^{-j_1 - \cdots - j_r},$$

where $|z_i| < |\nu(\nu + \lambda)| R_i$, $i = 1, \ldots, r,$

$$(2.19) \qquad G(z_1, \ldots, z_r : \nu, \lambda) = G[z_i; \lambda; 1; \nu; 1],$$

and $h(j_1, \ldots, j_r; z_1, \ldots, z_r; \lambda)$ *are given explicitly by* (2.21) *below.*

Further, for $j_1, \ldots, j_r \geqslant 1$, the coefficients, $h(j_1, \ldots, j_r; z_1, \ldots, z_r; \lambda)$, *can be expressed in terms of the derivatives of*

$$(2.20) \qquad h(z_1, \ldots, z_r) = h(0, \ldots, 0; z_1, \ldots, z_r; \lambda).$$

The proof of this theorem would run parallel to that of Theorem 1, and we omit details. Indeed, it is easily verified that

$$(2.21) \; h(j_1, \ldots, j_r; z_1, \ldots, z_r; \lambda) = \sum_{k_1,\ldots,k_r=0}^{\infty} b(k_1, \ldots, k_r) \prod_{i=1}^{r} c_{j_i, k_i}(\lambda) \frac{z_i^{k_i}}{(k_i!)^2},$$

where

$$(2.22) \qquad \begin{cases} c_{j_i+1, k_i}(\lambda) = \displaystyle\sum_{m_i=0}^{k_i-1} m_i(m_i + \lambda) c_{j_i, m_i}(\lambda), \\[2ex] c_{0, m_i}(\lambda) = 1, \quad m_i \geqslant 0, \\[2ex] c_{j_i+1, 0}(\lambda) = 0, \quad i = 1, \ldots, r. \end{cases}$$

If we write

$$(2.23) \quad B_{m_1,\ldots,m_r}(z_1,\ldots,z_r) = z_1^{m_1}\cdots z_r^{m_r}\frac{\partial^{m_1+\cdots+m_r}}{\partial z_1^{m_1}\cdots\partial z_r^{m_r}}\{h(z_1,\ldots,z_r)\},$$

where

$$h(z_1,\ldots,z_r) = h(0,\ldots,0;z_1,\ldots,z_r;\lambda)$$

$$(2.24)$$

$$= \sum_{k_1,\ldots,k_r=0}^{\infty} b(k_1,\ldots,k_r)\prod_{i=1}^{r}\frac{z_i^{k_i}}{(k_i!)^2},$$

then we have the first few coefficients given by

$$h(1,0,\ldots,0;z_1,\ldots,z_r;\lambda) = \frac{1}{3}B_{3,0,\ldots,0}(z_1,\ldots,z_r)$$

$$+ \frac{1}{2}(\lambda+1)B_{2,0,\ldots,0}(z_1,\ldots,z_r),$$

$$\vdots$$

$$h(0,\ldots,0,1;z_1,\ldots,z_r;\lambda) = \frac{1}{3}B_{0,\ldots,0,3}(z_1,\ldots,z_r)$$

$$+ \frac{1}{2}(\lambda+1)B_{0,\ldots,0,2}(z_1,\ldots,z_r);$$

$$h(2,0,\ldots,0;z_1,\ldots,z_r;\lambda) = \frac{1}{18}B_{6,0,\ldots,0}(z_1,\ldots,z_r)$$

$$+ \frac{1}{30}(5\lambda+17)B_{5,0,\ldots,0}(z_1,\ldots,z_r)$$

$$+ \frac{1}{8}(\lambda^2+8\lambda+11)B_{4,0,\ldots,0}(z_1,\ldots,z_r)$$

$$+ \frac{1}{3}(\lambda+1)(\lambda+2)B_{3,0,\ldots,0}(z_1,\ldots,z_r),$$

$$\vdots$$

$$h(0,\ldots,0,2;z_1,\ldots,z_r;\lambda) = \frac{1}{18}B_{0,\ldots,0,6}(z_1,\ldots,z_r)$$

$$+ \frac{1}{30}(5\lambda+17)B_{0,\ldots,0,5}(z_1,\ldots,z_r) + \frac{1}{8}(\lambda^2+8\lambda+11)B_{0,\ldots,0,4}(z_1,\ldots,z_r)$$

$$+ \frac{1}{3}(\lambda+1)(\lambda+2)B_{0,\ldots,0,3}(z_1,\ldots,z_r);$$

et cetera.

Remark 3. A substantial variation of Theorem 2 would follow if, for instance, we replace the product,

$$\prod_{i=1}^{r}\frac{(-\nu_i)_{\mu_i k_i}(\nu_i+\lambda_i)_{\rho_i k_i}}{(k_i!)^{\mu_i+\rho_i}}\left[-\frac{z_i}{\nu_i(\nu_i+\lambda_i)}\right]^{k_i},$$

in (2.17) by

$$
(-\nu)_{\mu_1 k_1 + \cdots + \mu_r k_r} (\nu + \lambda)_{\rho_1 k_1 + \cdots + \rho_r k_r} \prod_{i=1}^{r} \frac{[-z_i/\nu(\nu+\lambda)]^{k_i}}{(k_i!)^{\mu_i + \rho_i}},
$$

where, as before, $\mu_i, \rho_i > 0, i = 1, \ldots, r$.

Remark 4. If we let

$$(2.25) \qquad b(k_1, \ldots, k_r) = k_1! \cdots k_r! \, a(k_1, \ldots, k_r), \qquad k_i \geqslant 0, i = 1, \ldots, r,$$

then the following relationship holds between Theorems 1 and 2:

$$(2.26) \qquad \lim_{\lambda \to \infty} G(z_1, \ldots, z_r : \nu, \lambda) = F(z_1, \ldots, z_r : -\nu),$$

which is easy to verify.

3. **Asymptotic confluent expansions.** Our first result on asymptotic confluent expansions for functions of several variables is contained in

THEOREM 3. *Let*

$$(3.1) \qquad \sum_{k_1, \ldots, k_r = 0}^{\infty} c(k_1, \ldots, k_r) z_1^{k_1} \cdots z_r^{k_r} < \infty, \qquad |z_i| < R_i, i = 1, \ldots, r.$$

Then the series

$$(3.2) \qquad H[z_i; \beta_i; \mu_i] = \sum_{k_1, \ldots, k_r = 0}^{\infty} c(k_1, \ldots, k_r) \prod_{i=1}^{r} \frac{(\beta_i z_i)^{k_i}}{k_i! (\beta_i)_{\mu_i k_i}}$$

converges for all $z_i, \beta_i \neq 0, -1, -2, \ldots, \mu_i > 0, i = 1, \ldots, r$; *and the function,*

$$(3.3) \qquad H(z_1, \ldots, z_r : \beta) = H[z_i; \beta; 1],$$

possesses the asymptotic confluent expansion,

$$
H(z_1, \ldots, z_r : \beta) \sim \sum_{j_1, \ldots, j_r = 0}^{\infty} f(j_1, \ldots, j_r; z_1, \ldots, z_r)(-\beta)^{-j_1 - \cdots - j_r},
$$

$$(3.4) \qquad
\begin{aligned}
&|\beta| \to \infty, \; |\arg(\beta)| \leqslant \pi - \epsilon, 0 < \epsilon \leqslant \pi/2, \\
&|z_i| < R_i, i = 1, \ldots, r, \text{ if } |\arg(\beta)| \leqslant \pi/2; \\
&|z_i| < |\sin \arg(\beta)| R_i, i = 1, \ldots, r, \text{ if } \pi/2 \leqslant |\arg(\beta)| \leqslant \pi - \epsilon,
\end{aligned}
$$

where the coefficients $f(j_1, \ldots, j_r; z_1, \ldots, z_r)$ *are given explicitly by* (3.6) *below.*

Further, for $j_1, \ldots, j_r \geqslant 1$, *these coefficients can be expressed in terms of the derivatives of*

$$(3.5) \qquad f(z_1, \ldots, z_r) = f(0, \ldots, 0; z_1, \ldots, z_r).$$

Proof. The first part of this theorem stems from (3.1) and the test for convergence of multiple series [8, p. 157 *et seq.*]. The demonstration of the rest would make use of the Tricomi-Erdélyi formula (1.1) in a straightforward manner. Thus it is easily seen that

$$f(j_1, \ldots, j_r; z_1, \ldots, z_r)$$

$$(3.6) \qquad = \sum_{k_1, \ldots, k_r = 0}^{\infty} c(k_1, \ldots, k_r) \prod_{i=1}^{r} \frac{(k_i)_{j_i} z_i^{k_i}}{j_i!} B_{j_i}^{(1-k_i)},$$

and for $j_1, \ldots, j_r \geqslant 1$, we have

$$(3.7) \qquad f(j_1, \ldots, j_r; z_1, \ldots, z_r) = \sum_{m_1=0}^{2j_1-2} \cdots \sum_{m_r=0}^{2j_r-2} \prod_{i=1}^{r} \frac{z_i^{2+m_i} \psi(m_i)}{j_i! \, m_i!}$$

$$\cdot \frac{\partial^{2r+m_1+\cdots+m_r}}{\partial z_1^{2+m_1} \cdots \partial z_r^{2+m_r}} \{f(z_1, \ldots, z_r)\},$$

where, for convenience,

$$(3.8) \qquad \psi(m_i) = [\Delta_{k_i}^{m_i} \{(k_i)_{j_i} B_{j_i}^{(1-k_i)} / [k_i(k_i - 1)]\}]_{k_i=2} \qquad (i = 1, \ldots, r).$$

Remark 5. A slightly different form of Theorem 3 may be given by replacing the product,

$$\prod_{i=1}^{r} \frac{(\beta_i z_i)^{k_i}}{k_i! \, (\beta_i)_{\mu_i k_i}},$$

in (3.2) by

$$\{(\beta)_{\mu_1 k_1 + \cdots + \mu_r k_r}\}^{-1} \prod_{i=1}^{r} \frac{(\beta z_i)^{k_i}}{k_i!}, \qquad \mu_i > 0, i = 1, \ldots, r.$$

Remark 6. In terms of the derivatives,

$$(3.9) \qquad C_{m_1, \ldots, m_r}(z_1, \ldots, z_r) = z_1^{m_1} \cdots z_r^{m_r} \frac{\partial^{m_1+\cdots+m_r}}{\partial z_1^{m_1} \cdots \partial z_r^{m_r}} \{f(z_1, \ldots, z_r)\},$$

where

$$f(z_1, \ldots, z_r) = f(0, \ldots, 0; z_1, \ldots, z_r)$$

$$(3.10) \qquad = \sum_{k_1, \ldots, k_r = 0}^{\infty} c(k_1, \ldots, k_r) z_1^{k_1} \cdots z_r^{k_r},$$

the first few $f(j_1, \ldots, j_r; z_1, \ldots, z_r)$ are given as follows:

$$f(1, 0, \ldots, 0; z_1, \ldots, z_r) = \frac{1}{2} C_{2, 0, \ldots, 0}(z_1, \ldots, z_r),$$

$$\vdots$$

$$f(0, \ldots, 0, 1; z_1, \ldots, z_r) = \frac{1}{2} C_{0, \ldots, 0, 2}(z_1, \ldots, z_r);$$

$$f(1, \ldots, 1; z_1, \ldots, z_r) = \frac{1}{2^r} C_{2, \ldots, 2}(z_1, \ldots, z_r);$$

$$f(2, 0, \ldots, 0; z_1, \ldots, z_r) = \frac{1}{2} \, C_{2,0,\ldots,0}(z_1, \ldots, z_r)$$

$$+ \frac{2}{3} \, C_{3,0,\ldots,0}(z_1, \ldots, z_r) + \frac{1}{8} \, C_{4,0,\ldots,0}(z_1, \ldots, z_r),$$

$$\vdots$$

$$f(0, \ldots, 0, 2; z_1, \ldots, z_r) = \frac{1}{2} \, C_{0,\ldots,0,2}(z_1, \ldots, z_r)$$

$$+ \frac{2}{3} \, C_{0,\ldots,0,3}(z_1, \ldots, z_r) + \frac{1}{8} \, C_{0,\ldots,0,4}(z_1, \ldots, z_r);$$

et cetera.

For a nonnegative integer n, the multiple series defining the functions $F(z_1, \ldots, z_r : -n)$ and $G(z_1, \ldots, z_r : n, \lambda)$ in Theorems 1 and 2 would terminate and so these functions are well defined everywhere. Thus, under considerably weaker conditions than those stated earlier, Theorems 1 and 2 yield the asymptotic confluent expansions of $F(z_1, \ldots, z_r : -n)$ and $G(z_1, \ldots, z_r : n, \lambda)$, respectively, as $n \to \infty$. These immediate consequences of Theorems 1 and 2 are contained in

THEOREM 4. *Let*

$$(3.11) \qquad \sum_{k_1,\ldots,k_r=0}^{\infty} a(k_1, \ldots, k_r) \frac{z_1^{k_1}}{k_1!} \cdots \frac{z_r^{k_r}}{k_r!} < \infty, \qquad |z_i| < S_i, \; i = 1, \ldots, r;$$

$$(3.12) \qquad \sum_{k_1,\ldots,k_r=0}^{\infty} b(k_1, \ldots, k_r) \frac{z_1^{k_1}}{(k_1!)^2} \cdots \frac{z_r^{k_r}}{(k_r!)^2} < \infty, \qquad |z_i| < T_i, \; i = 1, \ldots, r.$$

Then, for an integer $n \geqslant 0$,

$$F(z_1, \ldots, z_r : -n) = \sum_{k_1,\ldots,k_r=0}^{n} a(k_1, \ldots, k_r) \prod_{i=1}^{r} \frac{(-n)_{k_i}}{k_i!} \left(-\frac{z_i}{n} \right)^{k_i}$$

$$(3.13)$$

$$\sim \sum_{j_1,\ldots,j_r=0}^{\infty} g(j_1, \ldots, j_r; z_1, \ldots, z_r) \, (-n)^{-j_1 - \cdots - j_r},$$

$$n \to \infty, \; |z_i| < S_i, \; i = 1, \ldots, r,$$

and

$$G(z_1, \ldots, z_r : n, \lambda)$$

$$(3.14) \qquad = \sum_{k_1,\ldots,k_r=0}^{n} b(k_1, \ldots, k_r) \prod_{i=1}^{r} \frac{(-n)_{k_i}(n+\lambda)_{k_i}}{(k_i!)^2} \left[-\frac{z_i}{n(n+\lambda)} \right]^{k_i}$$

$$\sim \sum_{j_1,\ldots,j_r=0}^{\infty} h(j_1, \ldots, j_r; z_1, \ldots, z_r; \lambda) [-n(n+\lambda)]^{-j_1 - \cdots - j_r},$$

$$n \to \infty, \; |z_i| < T_i, \; i = 1, \ldots, r,$$

where the coefficients $g(j_1, \ldots, j_r; z_1, \ldots, z_r)$ and $h(j_1, \ldots, j_r; z_1, \ldots, z_r; \lambda)$ are given explicitly by (2.9) and (2.21), respectively.

The following theorem provides yet another interesting asymptotic confluent expansion for a function of several variables.

THEOREM 5. *Let*

$$(3.15) \quad U(z_1, \ldots, z_r) = \sum_{k_1, \ldots, k_r = 0}^{\infty} d(k_1, \ldots, k_r) z_1^{k_1} \cdots z_r^{k_r} < \infty,$$

$$|z_i| < R_i, i = 1, \ldots, r.$$

Then

$$(3.16) \quad V[z_i; \mu_i; \sigma_i; a_i; b_i] = \sum_{k_1, \ldots, k_r = 0}^{\infty} d(k_1, \ldots, k_r) \prod_{i=1}^{r} \frac{(\sigma_i + a_i)_{\mu_i k_i} z_i^{k_i}}{(\sigma_i + b_i)_{\mu_i k_i}} < \infty,$$

provided $|z_i| < R_i$, $\mu_i > 0$, $\sigma_i + b_i \neq 0, -1, -2, \ldots, i = 1, \ldots, r$; *and for*
$\mu_1 = \cdots = \mu_r = 1$, *it can be rearranged when* $|z_i| < R_i/2, i = 1, \ldots, r$, *to yield the expansion,*

$$
\begin{aligned}
&V[z_i; 1; \sigma_i; a_i; b_i] \\
(3.17) \quad &= \sum_{j_1, \ldots, j_r = 0}^{\infty} w(j_1, \ldots, j_r; z_1, \ldots, z_r)[(\sigma_1 + b_1)_{j_1} \cdots (\sigma_r + b_r)_{j_r}]^{-1},
\end{aligned}
$$

where the coefficients $w(j_1, \ldots, j_r; z_1, \ldots, z_r)$ *are given explicitly by* (3.21) *below.*

However, *if the* a_i *and* b_i *are bounded, then* (3.17) *with* $\sigma_1 = \cdots = \sigma_r = \sigma$ *results in the asymptotic expansion,*

$$V(z_1, \ldots, z_r : \sigma) = V[z_i; 1; \sigma; a_i; b_i]$$

$$(3.18) \quad \sim \sum_{j_1, \ldots, j_r = 0}^{\infty} w(j_1, \ldots, j_r; z_1, \ldots, z_r) [(\sigma + b_1)_{j_1} \cdots (\sigma + b_r)_{j_r}]^{-1},$$

$$|\sigma| \to \infty, |\arg(\sigma + b_i)| \leq \pi - \delta_i, \delta_i > 0, |z_i| < R_i, i = 1, \ldots, r.$$

Moreover, if $[(\sigma + b_1)_{j_1} \cdots (\sigma + b_r)_{j_r}]^{-1}$ *is expanded in powers of* $1/\sigma$, (3.18) *can be written as an asymptotic confluent expansion in* $1/\sigma$.

Proof. The first assertion (3.16) would follow fairly readily as in Theorems 1–4. Also, since $\sigma_i + b_i \neq 0, -1, -2, \ldots, i = 1, \ldots, r$, from the familiar Vandermonde Theorem we obtain

$$(3.19) \quad \sum_{j_i=0}^{k_i} \frac{(-k_i)_{j_i} (b_i - a_i)_{j_i}}{j_i! (\sigma_i + b_i)_{j_i}} = \frac{(\sigma_i + a_i)_{k_i}}{(\sigma_i + b_i)_{k_i}}, \quad k_i \geq 0, i = 1, \ldots, r.$$

Making use of (3.15), we have

$$
\begin{aligned}
(3.20) \quad &\frac{\partial^{j_1 + \cdots + j_r}}{\partial z_1^{j_1} \cdots \partial z_r^{j_r}} \{U(z_1, \ldots, z_r)\} \\
&= \sum_{k_1, \ldots, k_r = 0}^{\infty} d(k_1, \ldots, k_r) \prod_{i=1}^{r} (-1)^{j_i} (-k_i)_{j_i} z_i^{k_i - j_i},
\end{aligned}
$$

which, in conjunction with (3.17) and (3.19), yields

$$w(j_1, \ldots, j_r; z_1, \ldots, z_r)$$

$$(3.21) \qquad = \prod_{i=1}^{r} \left\{ \frac{(b_i - a_i)_{j_i}(-z_i)^{j_i}}{j_i!} \right\} \frac{\partial^{j_1 + \cdots + j_r}}{\partial z_1^{j_1} \cdots \partial z_r^{j_r}} \{U(z_1, \ldots, z_r)\}.$$

If we assume that $|z_i| < X_i < R_i$, $i = 1, \ldots, r$, then the hypothesis (3.15) would evidently imply that

$$(3.22) \qquad d(k_1, \ldots, k_r) = O(X_1^{-k_1} \cdots X_r^{-k_r}), \qquad k_1, \ldots, k_r \to \infty.$$

From (3.17), (3.20) and (3.21) we thus observe that, for some constant $M > 0$,

$$|V[z_i; 1; \sigma_i; a_i; b_i]| \leqslant M \sum_{j_1, \ldots, j_r = 0}^{\infty} \left| \prod_{i=1}^{r} \frac{(b_i - a_i)_{j_i}}{(\sigma_i + b_i)_{j_i}} \right|$$

$$\cdot \sum_{k_1, \ldots, k_r = 0}^{\infty} \prod_{i=1}^{r} \binom{k_i + j_i}{j_i} \left(\frac{|z_i|}{X_i} \right)^{k_i + j_i}$$

$$= (1 - |z_1|/X_1)^{-1} \cdots (1 - |z_r|/X_r)^{-1} M$$

$$\cdot \sum_{j_1, \ldots, j_r = 0}^{\infty} \left| \prod_{i=1}^{r} \frac{(b_i - a_i)_{j_i}}{(\sigma_i + b_i)_{j_i}} \right| \left(\frac{X_1}{|z_1|} - 1 \right)^{-j_1} \cdots \left(\frac{X_r}{|z_r|} - 1 \right)^{-j_r},$$

which obviously converges when

$$(X_i/|z_i| - 1)^{-1} < 1, \qquad i = 1, \ldots, r,$$

that is, when

$$|z_i| < X_i/2 < R_i/2, \qquad i = 1, \ldots, r,$$

and the final assertion would follow from the fact that the $w(j_1, \ldots, j_r; z_1, \ldots, z_r)$ in (3.18) are identifiable as the Poincaré coefficients of $V(z_1, \ldots, z_r : \sigma)$ as $|\sigma| \to \infty$.

Remark 7. Another version of Theorem 5 can be given by further multiplying the product,

$$\prod_{i=1}^{r} \frac{(\sigma_i + a_i)_{\mu_i k_i} z_i^{k_i}}{(\sigma_i + b_i)_{\mu_i k_i}},$$

in (3.16) by such quotients as

$$(\sigma + a)_{\rho_1 k_1 + \cdots + \rho_r k_r} / (\sigma + b)_{\rho_1 k_1 + \cdots + \rho_r k_r}$$

times z_j^K, where $K = \Sigma_{i=1}^{r} k_i - k_j$, $1 \leqslant j \leqslant r$, $\sigma + b \neq 0, -1, -2, \ldots$, and $\rho_i > 0$, $i = 1, \ldots, r$.

In particular, for $\mu_i = \rho_i = 1$, $\sigma_i = \sigma$, $i = 1, \ldots, r$, and $j = 1$, we may state

THEOREM 6. *Let the hypothesis (3.15) of Theorem 5 be replaced by*

$$(3.23) \quad W(z_1, \ldots, z_r) = \sum_{k_1,\ldots,k_r=0}^{\infty} d(k_1, \ldots, k_r) z_1^{k_1}(z_1 z_2)^{k_2} \cdots (z_1 z_r)^{k_r} < \infty,$$

where $|z_1| < R_1$ and $|z_1 z_j| < R_j$, $j = 2, \ldots, r$.

Then the series

$$
\begin{aligned}
&Y(z_1, \ldots, z_r : \sigma) \\
(3.24) \quad &= \sum_{k_1,\ldots,k_r=0}^{\infty} d(k_1, \ldots, k_r) \frac{(\sigma + a_1)_{k_1+\cdots+k_r} \, \Pi_{j=2}^{r} (\sigma + a_j)_{k_j}}{(\sigma + b_1)_{k_1+\cdots+k_r} \, \Pi_{j=2}^{r} (\sigma + b_j)_{k_j}} \\
&\quad \cdot z_1^{k_1}(z_1 z_2)^{k_2} \cdots (z_1 z_r)^{k_r},
\end{aligned}
$$

converges for $|z_1| < R_1$, $|z_1 z_j| < R_j$, $j = 2, \ldots, r$, $\sigma + b_i \neq 0, -1, -2, \ldots, i = 1, \ldots, r$; and it can be rearranged when $|z_1| < \epsilon_1 R_1$, $|z_1 z_j| < \epsilon_j R_j$, $j = 2, \ldots, r$, for some ϵ_i, $0 < \epsilon_i < 1$, $i = 1, \ldots, r$, to yield the expansion,

$$
\begin{aligned}
&Y(z_1, \ldots, z_r : \sigma) \\
(3.25) \quad &= \sum_{j_1,\ldots,j_r=0}^{\infty} u(j_1, \ldots, j_r; z_1, \ldots, z_r) \left[(\sigma + b_1)_{j_1} \cdots (\sigma + b_r)_{j_r} \right]^{-1},
\end{aligned}
$$

where

$$
\begin{aligned}
&u(j_1, \ldots, j_r; z_1, \ldots, z_r) \\
(3.26) \quad &= \prod_{i=1}^{r} \left\{ \frac{(b_i - a_i)_{j_i} (-z_i)^{j_i}}{j_i!} \right\} \frac{\partial^{j_1+\cdots+j_r}}{\partial z_1^{j_1} \cdots \partial z_r^{j_r}} \{ W(z_1, \ldots, z_r) \}.
\end{aligned}
$$

Moreover, for bounded a_i and b_i, (3.26) holds asymptotically as

$$
\begin{aligned}
&Y(z_1, \ldots, z_r : \sigma) \\
(3.27) \quad &\sim \sum_{j_1,\ldots,j_r=0}^{\infty} u(j_1, \ldots, j_r; z_1, \ldots, z_r) \left[(\sigma + b_1)_{j_1} \cdots (\sigma + b_r)_{j_r} \right]^{-1},
\end{aligned}
$$

$|\sigma| \to \infty$, $|\arg(\sigma + b_i)| \leq \pi - \delta_i$, $\delta_i > 0$, $i = 1, \ldots, r$, $|z_1| < R_1$, $|z_1 z_j| < R_j$, $j = 2, \ldots, r$, which leads to an asymptotic confluent expansion in $1/\sigma$ if $\left[(\sigma + b_1)_{j_1} \cdots (\sigma + b_r)_{j_r} \right]^{-1}$ is expanded in reciprocal powers of σ.

Remark 8. For $r = 2$, this last theorem would evidently reduce to the main result (Theorem 1, p. 605) in Deshpande's paper [4].

4. Applications. By assigning suitable values to the arbitrary coefficients $a(k_1, \ldots, k_r)$, $b(k_1, \ldots, k_r)$, $c(k_1, \ldots, k_r)$ and $d(k_1, \ldots, k_r)$, $k_i \geqslant 0$, $i = 1, \ldots, r$, Theorems 1–6 can be applied to deduce analytic or asymptotic confluent expansions for various special functions of several variables; e.g., the Lauricella hypergeometric functions $F_A^{(r)}$, $F_B^{(r)}$, $F_C^{(r)}$ and $F_D^{(r)}$ of r variables and their generalizations studied by Srivastava and Daoust ([7], [8]). As an illustration, we apply Theorems 1 and 4 to derive an asymptotic confluent expansion for a certain generalization of the Lauricella functions, defined by (cf. [7, p. 454]; see also [8, p. 158])

$$F\begin{matrix} p:q_1;\ldots;q_r \\ P:Q_1;\ldots;Q_r \end{matrix}\left(\begin{matrix} z_1 \\ \vdots \\ z_r \end{matrix}\right) \equiv F\begin{matrix} p:q_1;\ldots;q_r \\ P:Q_1;\ldots;Q_r \end{matrix}\left(\begin{matrix} (\alpha_p): (\beta'_{q_1});\ldots;(\beta^{(r)}_{q_r}); \\ (\gamma_P): (\delta'_{Q_1});\ldots;(\delta^{(r)}_{Q_r}); \end{matrix} z_1,\ldots,z_r\right)$$

(4.1)
$$= \sum_{k_1,\ldots,k_r=0}^{\infty} \Lambda(k_1,\ldots,k_r) \prod_{i=1}^{r} \frac{z_i^{k_i}}{k_i!},$$

where, and in what follows,

(4.2) $\quad \Lambda(k_1^{\bullet},\ldots,k_r) = \dfrac{\Pi_{j=1}^{p}\,(\alpha_j)_{k_1+\cdots+k_r}\;\Pi_{j=1}^{q_1}\,(\beta'_j)_{k_1}\;\cdots\;\Pi_{j=1}^{q_r}\,(\beta^{(r)}_j)_{k_r}}{\Pi_{j=1}^{P}\,(\gamma_j)_{k_1+\cdots+k_r}\;\Pi_{j=1}^{Q_1}\,(\delta'_j)_{k_1}\;\cdots\;\Pi_{j=1}^{Q_r}\,(\delta^{(r)}_j)_{k_r}},$

and for convergence of the multiple series in (4.1),

(4.3) $\qquad\qquad 1 + P + Q_i - p - q_i \geq 0, \quad i = 1,\ldots,r;$

the equality holds when, in addition,

(4.4) $\qquad\qquad p > P \quad \text{and} \quad |z_1|^{1/(p-P)} + \cdots + |z_r|^{1/(p-P)} < 1,$

or

(4.5) $\qquad\qquad p \leq P \quad \text{and} \quad \max\{|z_1|,\ldots,|z_r|\} < 1.$

Thus, if we put
(4.6)
$$D_{m_1,\ldots,m_r}(z_1,\ldots,z_r) = z_1^{m_1}\cdots z_r^{m_r}\,\Lambda(m_1,\ldots,m_r)$$

$$\cdot F\begin{matrix} p:q_1;\ldots;q_r \\ P:Q_1;\ldots;Q_r \end{matrix}\left(\begin{matrix} (\alpha_p)+m_1+\cdots+m_r: (\beta'_{q_1})+m_1;\ldots;(\beta^{(r)}_{q_r})+m_r; \\ (\gamma_P)+m_1+\cdots+m_r: (\delta'_{Q_1})+m_1;\ldots;(\delta^{(r)}_{Q_r})+m_r; \end{matrix} z_1,\ldots,z_r\right)$$

and let n be a nonnegative integer, we obtain the desired asymptotic expansion,

$$F\begin{matrix} p: 1+q_1;\ldots;1+q_r \\ P:\quad Q_1;\ldots;\quad Q_r \end{matrix}\left(\begin{matrix} (\alpha_p): -n,(\beta'_{q_1});\ldots;-n,(\beta^{(r)}_{q_r}); \\ (\gamma_P):\quad (\delta'_{Q_1});\ldots;\quad (\delta^{(r)}_{Q_r}); \end{matrix} -\frac{z_1}{n},\ldots,-\frac{z_r}{n}\right)$$

$$\sim F\begin{matrix} p:q_1;\ldots;q_r \\ P:Q_1;\ldots;Q_r \end{matrix}\left(\begin{matrix} z_1 \\ \vdots \\ z_r \end{matrix}\right)$$

(4.7)
$$-\frac{1}{2n}\,[D_{2,0,\ldots,0}(z_1,\ldots,z_r) + \cdots + D_{0,\ldots,0,2}(z_1,\ldots,z_r)]$$

$$+ O(1/n^2), \quad n \to \infty,$$

provided that (4.3) and (4.4), or (4.3) and (4.5), hold.

For special values of the nonnegative integers p, q_i, P and Q_i, $i = 1,\ldots,r$, (4.7) reduces to the corresponding asymptotic expansions for the Lauricella functions except possibly the function $F_C^{(r)}$ of the third type. In particular, for the fourth

Lauricella function $F_D^{(r)}$, we have from (4.7) with $p = P = 1, q_i = Q_i = 0, i = 1,$ $\ldots, r$,

$$F_D^{(r)}[\alpha, -n, \ldots, -n; \gamma; -z_1/n, \ldots, -z_r/n] \sim {}_1F_1[\alpha; \gamma; z_1 + \cdots + z_r]$$

(4.8)
$$- \frac{\alpha(\alpha + 1)(z_1^2 + \cdots + z_r^2)}{2n\gamma(\gamma + 1)} {}_1F_1[\alpha + 2; \gamma + 2; z_1 + \cdots + z_r]$$

$$+ O(n^{-2}), \quad n \to \infty,$$

where ${}_1F_1[\alpha; \gamma; z]$ is the familiar confluent hypergeometric function.

Evidently, this last asymptotic expansion (4.8) would hold for all $z_1, \ldots, z_r$.

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